and for example,

$$v_1-v_0+v_2-v_1+\ldots.v_\lambda-v_{\lambda-1}=v_\lambda-v_0=u_1-u_0$$
$$=\delta v_0+\delta v_1+\ldots.\delta_{\lambda-1}=\triangle u_0 \ldots\ldots\ldots\ldots\ldots(5),$$

and similarly for the other Δu's. As the latter are the sum of λ terms of the δv's, we have by the formula for such a sum,

$$\Delta u_0=\binom{\lambda}{1}\delta v_0+\binom{\lambda}{2}\delta^2 v_0+\ldots.\binom{\lambda}{\lambda}\delta^\lambda v_0.$$

$$\Delta u_1=\binom{\lambda}{1}\delta v_\lambda+\binom{\lambda}{2}\delta^2 v_\lambda+\ldots.\binom{\lambda}{\lambda}\delta^\lambda v_\lambda.$$

. . . . . .

$$\Delta u_r=\binom{\lambda}{1}\delta v_{r\lambda}+\binom{\lambda}{2}\delta^2 v_{r\lambda}+\ldots.\binom{\lambda}{\lambda}\delta^\lambda v_{r\lambda}.$$

. . . . . (6).

$$\Delta^2 u_0=\binom{\lambda}{1}(\delta v_\sigma-\delta v_0)+\binom{\lambda}{2}(\delta^2 v_\lambda-\delta^2 v_0)+\ldots\binom{\lambda}{\lambda}(\delta^\lambda v_\lambda-\delta^\lambda v_0).$$

. . . . . .

$$\Delta^{r+1}u_0=\binom{\lambda}{1}(\delta v_\lambda-\delta v_0)_r+\binom{\lambda}{2}(\delta^2 v_\lambda-\delta^2 v_0)_r+\ldots.\binom{\lambda}{\lambda}(\delta^\lambda v_\lambda-\delta^\lambda v_0)_r$$

where $(\delta^\kappa v_\lambda-\delta^\kappa v_0)_r=\delta^\kappa v_{r\lambda}-\binom{r}{1}\delta^\kappa v_{(r-1)\lambda}+\binom{r}{2}\delta^\kappa v_{(r-2)\lambda}-\ldots.$

Expressing $v_{r\lambda}$, $v_{(r-1)\lambda}$, ... etc., in terms of v_0,

$$\Delta^{r+1}u_0=\binom{\lambda}{1}(\Delta^r(0,r)\delta^{r+1}v_0+\Delta^r(0,r+1)\delta^{r+2}v_0+\ldots.)$$
$$+\binom{\lambda}{2}(\Delta^r(0,r)\delta^{r+2}v_0+\ldots)+\ldots.(7)$$

where
$$(x,H)=\binom{x\lambda}{H},\ \Delta^r(0,H)=\binom{r\lambda}{H}-\binom{r}{1}\binom{(r-1)\lambda}{H}+\ldots.(-1)^{r-1}\binom{r}{r-1}\binom{\lambda}{H},$$

and all the coefficients of the δv's in (7) of lower order than the $(r+1)$st are zero by means of the relation

$$\Delta^r(x-r,H)=0, \qquad (8)$$

if $r>H$. This is seen by considering the fact that

$$\left[\begin{matrix}(x-r)\lambda \\ H\end{matrix}\right]=\frac{(x-r)\lambda(x-r)\lambda-1)\ldots.((x-r)\lambda-H+1)}{H\,!}$$

is of degree H in x; therefore $\Delta^r(x-r,\ H)$ is of degree $H-r$ in x, and therefore zero, if $r>H$. We may also notice that,

$$\Delta^r(0,\ r)=A_r\lambda^r=\lambda^r,$$

$$\Delta^r(0,\ r+1)=A_{r+1}\lambda^r(\lambda-1)=\frac{r}{2}\lambda^r(\lambda-1), \tag{9}$$

where A_r, and A_{r+1} are the coefficients of x^r and x^{r+1} in $(e^x-1)^r$.* From this

$$\Delta^{r+1}u_0=\lambda^{r+1}\delta^{r+1}v_0+\frac{r}{2}\lambda^{r+1}(\lambda-1)\delta^{r+2}v_0+\ldots. \tag{10}$$

that is, each $\Delta^\kappa\ u_0$ is expressed in terms of $\lambda^\kappa\ \delta^\kappa\ v_0$, and higher differences than those of order κ in the δv's. We see:

If the κth order of differences in the δv's is constant, $\delta^{\kappa+r}v_0=0$. and $\Delta^\kappa\ u_0=\lambda^\kappa\ \delta^\kappa\ v_0$ exactly and $\Delta^{\kappa+r}u_0=0$, and conversely if $\Delta^{\kappa+r}u_0=0$, it follows that $\delta^{\kappa+r}v_0=0$. and $\Delta^\kappa\ u_0=\lambda^\kappa\ \delta^\kappa\ v_0$, for a series of linear functions of the δv's beginning with the $(\kappa+1)$st order of differences will be zero independently of λ. Hence the order of constancy is the same for both series.

§3. The Series $(1+\Delta)^{\nu\ \lambda}f(x)$.

The series, $(\nu^1=\frac{\nu}{\lambda})$,

$$u_0+\left[\begin{matrix}\nu^1\\1\end{matrix}\right]\Delta u_0+\left[\begin{matrix}\nu^1\\2\end{matrix}\right]\Delta^2 u_0+\ldots.\left[\begin{matrix}\nu^1\\r\end{matrix}\right]\Delta^r u_0+\ldots. \tag{11}$$

within the limits of convergence, is taken as starting point in the proof. By convention, this development may be neatly expressed symbolically as $(1+\Delta)^{\nu/\lambda}u_0$, if it is agreed to understand that only the simplest expansion of the operator is taken, and not any one of the values it might have from its similarity to a fractional power of a real binomial $1+x$. This however will be mere notation and does not form any necessary part of the proof.

§4. The Convergency of the Series.

There are various tests of convergency of series. One of the most elementary gives the following result: If beginning with and after a value r,

$$\operatorname{mod}\frac{\nu^1-r+1}{r}\ \frac{\Delta^r u_0}{\Delta^{r-1}u_0}<\kappa<1, \tag{12}$$

*See "Note on Integral and Integro-Geometric Series" by the author, *Annals of Mathematics*, October, 1897, page 184.

the series is convergent. The series is certainly convergent at least :

1. If the κth order of differences of the Δu's is constant; then $\Delta^{\kappa+1}u_0 = \Delta^{\kappa+2}u_0 = \ldots = 0$, and the series breaks off and is a finite series.

2. If

$$2 > \operatorname{Lim} \frac{\Delta^{r-1}u_1}{\Delta^{r-1}u_0} > 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13),$$

for, since $\dfrac{\Delta^r u_0}{\Delta^{r-1}u_0} = \dfrac{\Delta^{r-1}u_1 - \Delta^{r-1}u_0}{\Delta^{r-1}u_0} = \dfrac{\Delta^{r-1}u_1}{\Delta^{r-1}u_0} - 1$, and

$$-1 < \operatorname{Lim} \frac{\Delta^r u_0}{\Delta^{r-1}u_0} < 1,$$

the condition of convergency is satisfied. It is seen that Rice's two cases are contained here, the first in the first, and the second in the second condition. It was not the object here to go into the question of how far the series is convergent, but only to point out the possibility of its convergence. The fuller meaning of the second condition, and the possibility of farther conditions by finer tests might form a subject of future investigation.

§5. The Δu Series Expressed in Terms of the δv's.

Expressing the series $u_0 + \binom{\nu^1}{1}\Delta u_0 + \binom{\nu^1}{2}\Delta^2 u_0 + \ldots + \binom{\nu^1}{r}\Delta^r u_0 + \ldots$ in terms of the δv's by means of (7), the collected coefficient c_r of $\delta^r v_0$ becomes

$$c_r = \sum_{\kappa=0}^{\kappa=r-1} \binom{\kappa+1}{\nu^1} \sum_{\mu=1}^{\mu=r-\kappa} \binom{\lambda}{\mu} \Delta^\kappa (0, r-\mu) \quad \ldots\ldots\ldots\ldots (14).$$

By means of the relations

$$\binom{\rho+\sigma}{r} = \binom{\rho}{0}\binom{\sigma}{r} + \binom{\rho}{1}\binom{\sigma}{r-1} + \ldots + \binom{\rho}{r-1}\binom{\sigma}{1} + \binom{\rho}{r}\binom{\sigma}{0} \quad \ldots\ldots\ldots (15)$$

and $\Delta^\kappa (0, \sigma) = 0$, where $\sigma < \kappa$, it can be reduced to

$$c_r = \sum_{\kappa=1}^{\kappa=r} \binom{\nu^1}{\kappa} \Delta^\kappa (0, r) \quad \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

$\Delta^\kappa (0, r)$ is of the form $a_0\lambda^r + a_1\lambda^{r-1} + \ldots + a_{r-\kappa}\lambda^\kappa$. $\binom{\nu^1}{\kappa}\Delta^\kappa (0, r)$ is therefore an integral function of λ and ν, of degree $r-1$ in λ, and of degree κ in ν; and c_r is integral and of no higher degree than $r-1$ in λ, and of degree r in ν. We

shall show that it does not contain λ at all. $\binom{\kappa\lambda}{r}$ is an integral function of $\kappa\lambda$ of degree r, and is divisible by $\kappa\lambda$; it is of the form $(\kappa\lambda)^r+a_1(\kappa\lambda)^{r-1}+\ldots.a_{r-1}(\kappa\lambda)$ $=\kappa^r\lambda^r+\kappa^{r-1}a_1\lambda^{r-1}+\ldots.\kappa a_{r-1}\lambda$; $\binom{(\kappa-1)\lambda}{r}$ is therefore of the form $(\kappa-1)^r\lambda^r$ $+(\kappa-1)^{r-1}a_1\lambda^{r-1}+\ldots.(\kappa-1)a_{r-1}\lambda$. Hence the coefficient of λ^σ in $\Delta^\kappa(0, r)$ is

$$a_{r-\sigma}\frac{\kappa^\sigma-\kappa(\kappa-1)^\sigma+\frac{\kappa(\kappa-1)}{2!}(\kappa-2)^\sigma-\ldots.(-1)^{\kappa-1}\kappa.1^\sigma}{r!}$$

$$=\frac{1}{r(r-1)\ldots.(\sigma+1)}a_{r-\sigma}A_{(\kappa)+\sigma-\kappa}$$

where $A_{(\kappa)+\sigma-\kappa}$ is the coefficient of $x^{\sigma-\kappa}$ in $(e^x-1)^\kappa$, and where this is zero if $\sigma<\kappa$,* which proves the statement that $\Delta^\kappa(0, r)$ contains the factor λ^κ.

§6. The Coefficient c_r of $\delta^r v_0$.

We have shown that c_r is of the form,

$$c_r=f_r(\nu, r)+f_{r,}(\nu, r)\lambda+\ldots.f_1(\nu, r)\lambda^{r-1}\ldots\ldots\ldots\ldots\ldots(17),$$

the indices denoting the degree in ν; we shall prove that

$$f_\kappa(\nu, r)\equiv 0,\ \kappa=1, 2, \ldots.r-1, \text{ and } c_r\equiv f_r(\nu, r)\equiv\binom{\nu}{r}\quad\ldots\ldots\ldots(18).$$

The equation $c_r-\binom{\nu}{r}=0$, or

$$f_1(\nu, r)\lambda^{r-1}+f_2(\nu, r)\lambda^{r-2}+\ldots.f_r(\nu, r)-\binom{\nu}{r}=0\quad\ldots\ldots\ldots(19),$$

is satisfied by r values of λ,

$$\nu=\kappa\lambda,\ \lambda=\frac{\nu}{\kappa},\ \nu^1=\kappa,\ \kappa=1, 2, \ldots.r\ \ldots\ldots\ldots\ldots\ldots(20).$$

We have when $\nu^1=\kappa$,

*See "Note on Integral and Integro-Geometric Series," l. c.

$$c_r = \binom{\varkappa}{1}\Delta(0,r) + \binom{\varkappa}{2}\Delta^2(0,r) = \ldots\ldots \binom{\varkappa}{\varkappa}\Delta^{\varkappa}(0,r) \ldots\ldots\ldots (21).$$

The terms beyond that containing $\binom{\nu^1}{\varkappa} = \binom{\varkappa}{\varkappa}$ in c_r all vanish because they contain the factor $(\nu - \lambda\varkappa) = 0$. The coefficient of $\binom{\frac{\rho\nu}{\varkappa}}{r}$ in c_r is, $(\rho < \varkappa)$,

$$\binom{\varkappa}{\rho} - \binom{\varkappa}{\rho+1}\binom{\rho+1}{1} + \binom{\varkappa}{\rho+2}\binom{\rho+2}{2} - \ldots\ldots (-1)^{\sigma}\binom{\varkappa}{\rho+\sigma}\binom{\rho+\sigma}{\sigma}$$
$$+ \ldots\ldots (-1)^{\varkappa-\rho}\binom{\varkappa}{\varkappa}\binom{\varkappa}{\varkappa-\rho} \ldots\ldots\ldots\ldots\ldots\ldots (22).$$

The general term of this is $(-1)^{\sigma}\binom{\varkappa}{\rho+\sigma}\binom{\rho+\sigma}{\sigma} = (-1)^{\sigma}\binom{\varkappa}{\rho}\binom{\varkappa-\rho}{\sigma}$ and (22) becomes

$$\binom{\varkappa}{\rho}\sum_{\sigma=0}^{\sigma=\varkappa-\rho}(-1)^{\sigma}\binom{\varkappa-\rho}{\sigma} = \binom{\varkappa}{\rho}(1-1)^{\varkappa-\rho} = 0 \ldots\ldots\ldots\ldots\ldots (23),$$

and this is true for all values of ρ, from $\rho = 1$, to $\rho = \varkappa - 1$; only the single value $\binom{\varkappa}{\varkappa}\binom{\nu}{r} = \binom{\nu}{r}$ of c_r is left, and

$$c_r = \left[\begin{matrix}\nu\\ r\end{matrix}\right], \text{ for } \lambda = \frac{\nu}{\varkappa},\ \varkappa = 1, 2, \ldots . r \ldots\ldots\ldots\ldots\ldots (24),$$

hence (19) is an identity, and

$$c_r \equiv \left[\begin{matrix}\nu\\ r\end{matrix}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25).$$

§7. The Sum of the Series.

By means of the result of §6, we can express the sum of the series (11), viz:

$$u_0 + \left[\begin{matrix}\nu^1\\ 1\end{matrix}\right]\Delta u_0 + \left[\begin{matrix}\nu^1\\ 2\end{matrix}\right]\Delta^2 u_0 + \ldots\ldots \left[\begin{matrix}\nu^1\\ r\end{matrix}\right]\Delta^r u_0 + \ldots\ldots = v_0 + \left[\begin{matrix}\nu\\ 1\end{matrix}\right]\delta v_0$$
$$+ \ldots\ldots + \ldots\ldots \left[\begin{matrix}\nu^1\\ r\end{matrix}\right]\delta^r v_0 + \ldots\ldots \left[\begin{matrix}\nu\\ \nu\end{matrix}\right]\delta^{\nu} v_0 = v_{\nu} = f(x + \tfrac{\nu}{\lambda}), \text{ or}$$

$$f(x+\tfrac{\nu}{\lambda}) = f(x) + \frac{\nu}{\lambda}\Delta f(x) + \frac{\frac{\nu}{\lambda}\left[\frac{\nu}{\lambda} - 1\right]}{2!}\Delta^2 f(x)$$
$$+ \ldots\ldots \frac{\frac{\nu}{\lambda}\left[\frac{\nu}{\lambda} - 1\right]\ldots\ldots\left[\frac{\nu}{\lambda} - r + 1\right]}{r!}\Delta^r f(x) + \ldots\ldots\ldots (26).$$

If we use the symbolic abbreviation before suggested with the understanding that by it we mean only the right member of (26), then we have proved,

$$f(x + \tfrac{\nu}{\lambda}) = (1+\Delta)^{\nu\lambda} f(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (27).$$

§8. The Formula (27) Proved by Symbolic Methods.

Immediately after the writer had found the previous non-symbolic proof that the formula $f(x+y)=(1+\Delta)^y f(x)$ holds for a commensurable fractional y, he discovered by symbolic methods a form of proof which is the opposite extreme of simplicity as compared with the non-symbolic. The two proofs confirm each other. We have:

$$\Delta u_0=((1+\delta)^\lambda-1)v_0.$$
$$\Delta u_1=((1+\delta)^\lambda-1)(1+\delta)^\lambda v_0.$$
$$\Delta u_2=((1+\delta)^\lambda-1)(1+\delta)^{2\lambda}v_0.$$
$$\cdots\cdots\cdots\cdots$$
$$\Delta u_r=((1+\delta)^\lambda-1)(1+\delta)^{r\lambda}v_0.$$
$$\cdots\cdots\cdots\cdots$$
$$\Delta^2 u_0=((1+\delta)^\lambda-1)^2 v_0.$$
$$\Delta^3 u_0=((1+\delta)^\lambda-1)^3 v_0.$$
$$\cdots\cdots\cdots\cdots$$
$$\Delta^r u_0=((1+\delta)^\lambda-1)^r v_0. \qquad (28).$$

Hence by the use of the formulas, (28), and confining the meaning of the symbol $(1+\Delta)^{\nu/\lambda}$ as before,

$$(1+\Delta)^{\nu/\lambda}u_0=u_0+\binom{\nu}{1}\Delta u_0+\ \cdots\binom{\nu}{r}\Delta^r u_0+\cdots.$$

$$=v_0+\binom{\nu}{1}((1+\delta)^\lambda-1)v_0+\ \cdots\binom{\nu}{r}((1+\delta)^\lambda-1)^r v_0+\cdots.$$

$$=(1+(1+\delta)^\lambda-1)^{\nu/\lambda}v_0=(1+\delta)^{\nu/\lambda\cdot\lambda}v_0=(1+\delta)^\nu v_0=v_\nu=f(x+\frac{\nu}{\lambda}),$$

or $f(x+\frac{\nu}{\lambda})=(1+\Delta)^{\nu/\lambda}f(x)$.

§9. Extension to Negative Exponents.

Confining the symbols as before, and the series to convergency, and n and r to positive integers, $n>r$,

$$f(x+(n-r))=(1+\Delta)^{n-r}f(x)=f((x+n)-r)\cdots\cdots\cdots\cdots(29).$$

Also $(1+\Delta)^{-r}f(x+n)=f(x+n)-r\Delta f(x+n)+\cdots.$

$$=(1+\Delta)^n(f(x)-r\Delta f(x)+\cdots.)$$
$$=(1+\Delta)^n(1+\Delta)^{-r}f(x)$$
$$=(1+\Delta)^{n-r}f(x) \qquad (30).$$

Comparing (29) and (30) we have

$$f((x+n)-r)=(1+\Delta)^{-r}f(x+n)\ldots\ldots\ldots\ldots\ldots\ldots(31),$$

or denoting $(x+n)$ by a single letter,

$$f(x-r)=(1+\Delta)^{-r}f(x)\ldots\ldots\ldots\ldots\ldots\ldots(32).$$

In order to extend to a negative fractional commensurable index, we choose, $-\nu=\varkappa\lambda$, $\varkappa=1, 2, \ldots .r$, in §6, and come out with $c_r=\binom{-\nu}{r}$ always, where we are supposed to have found the value of the simplest series $(1+\Delta)^{-\nu/\lambda}f(x)$ in terms of δv's, and thus we have

$$(1+\Delta)^{-\nu/\lambda}f(x)=(1+\delta)^{-\nu}f(x)\ldots\ldots\ldots\ldots\ldots\ldots(33).$$

Or as in §8, by symbolic operations we arrive at

$$(1+\Delta)^{-\nu^1}u_0=(1+\delta)^{-\nu^1\lambda}v_0=(1+\delta)^{-\nu}v_0.$$

By reasoning analogous to that by which (32) was found,

$$(1+\delta)^{-\nu}f(x)=f(x-\frac{\nu}{\lambda})\ \ldots\ldots\ \ldots\ldots\ldots\ \ldots(34).$$

Comparing (33) and (34),

$$f(x-\frac{\nu}{\lambda})=(1+\Delta)^{-\nu/\lambda}f(x)\ \ldots\ldots\ldots\ldots\ldots\ldots(35).$$

§10. Extension to an Incommensurable y.

Here we have to use limits. Let y be incommensurable, y^1 commensurable, and let $y^1\doteq y$. Then by a combination of the previous paragraphs,

$$f(x+y^1)=(1+\Delta)^{y^1}f(x), \text{ and}$$

$$\operatorname*{Lim}_{y^1\doteq y} f(x+y)^1)=\operatorname*{Lim}_{y^1\doteq y}(1+\Delta)^{y^1}f(x), \text{ or}$$

$$f(x+y)=(1+\Delta)^{y}f(x)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36).$$

Hence within the limits of convergence, and for the confined meaning of the symbol $(1+\Delta)^y$, we have proved formula (36) to hold for all real values whatsoever of y.

Syracuse University, 1 December, 1900.

TWO HYDRAULIC METHODS TO EXTRACT THE nth ROOT OF ANY NUMBER.

By DR. ARNOLD EMCH, University of Colorado.

1. The first method is based upon the construction of a vessel which is bounded by such a surface of revolution that the weight of the water displaced by the surface is always equal to the nth power of the depth of immersion. Suppose that $x=f(y)$ is the equation of a meridian of the surface, Fig. I, for which OY is the axis of revolution, and that the meridian touches the line OX at O. Using one foot as the unit of measure and designating by w the weight of a cubic foot of water which may be assumed as 62.5 pounds, the weight of the displaced water, when the depth of immersion is y, is

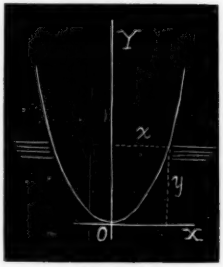


Fig. 1.

$$W=\pi.w\int_0^y x^2.dy=\pi.w\int_0^y f^2(y)dy \quad \ldots.(1),$$

and this shall equal y^n. Hence

$$\pi w\int_0^y f^2(y)dy=y^n \ldots.(2),$$

from which $x=f(y)$ is easily found:

$$x=f(y)=\sqrt{\frac{n}{\pi.w}}y^{\frac{1}{2}(n-1)} \ldots.(3), \text{ or } y=\sqrt{\frac{\pi.w}{n}}x^{2/(n-1)} \ldots.(4).$$

Giving n successively the values 1, 2, 3, ... we find for the equations of the meridians

$$x=\sqrt{\frac{1}{\pi.w}}=0.05052\ldots.(5),$$

$$y=\sqrt{\frac{\pi.w}{2}}x^2=9.9083.x^2\ldots.(6),$$

$$y=\sqrt{\frac{\pi.w}{3}}x=8.0901.x\ldots.(7),$$

etc., (unit=1 foot)

so that for the first, square, and cube roots, the surfaces are respectively a cylin-

der, a paraboloid and a cone of rotation. Now by the principle of Archimedes the weight of the water displaced is equal to the weight of the vessel. Hence, if this weight is Q (in pounds) and if we want to extract the nth root of a given number N, we add to Q a weight P in such a manner that $Q+P=N$. After the vessel is in equilibrium, there is

$$Q+P=N=W=y^n$$

and consequently,

$$y=\sqrt[n]{N} \ldots . (8).$$

The depth of immersion, y, is therefore the nth root of the number N.

In Fig. 2 an apparatus for the extraction of the square root has been sketched. The shape of the paraboloid of rotation is determined by equation (6) and corresponds to the wetted surface of the vessel. Suppose P is the weight to be added at A to make $Q+P=N$, then the difference of the water-levels before and after immersion will be equal to y or the square root of N. The difference in levels may be measured by the hook-guage G. In the mechanical execution of this apparatus it is not difficult to secure any reasonable precision and to eliminate errors of construction and observation.

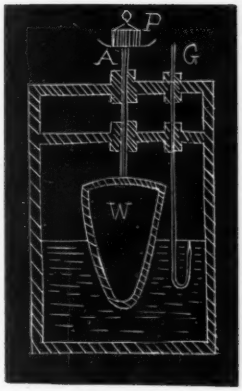


Fig. 2.

2. A second method to extract the nth root of any number may be based upon the problem to find the time necessary of emptying a vessel of given form through a small orifice at the bottom. We assume again a surface of revolution and refer it to the same axes as in the previous problem. Then the time dt to lower the level of the water in the vessel by dy units is expressed by the formula,

$$dt=\frac{\pi . x^2 . dy}{a\sqrt{(2gy)}}=\frac{\pi f^2(y) dy}{a\sqrt{(2gy)}} * \ldots . (9),$$

where a is the cross-section of the orifice.

Hence the time of emptying the vessel from the original level y is

$$t=\frac{\pi}{a\sqrt{(2g)}} \int_y^0 \frac{f^2(y) dy}{\sqrt{y}} \ldots (10).$$

We shall now determine $f(y)$ in such a manner that

$$t=\sqrt[n]{y}=\frac{\pi}{a\sqrt{(2g)}} \int_y^0 \frac{f^2(y) dy}{\sqrt{y}} \ldots . (11).$$

*See Merriman's Hydraulics, page 56.

From (11) we find for $x=f(y)$ the expression

$$x=\sqrt{\frac{a\sqrt{(2g)}}{n.\pi}}.y^{(2-n)/4n}\ldots.(12),$$

which for $n=1, 2, 3, \ldots.$ *i. e.*, for the first, square, and cube-root, becomes

$$x=\sqrt{\frac{a\sqrt{(2g)}}{\pi}}.y^{\frac{1}{4}}\ldots.(13),$$

$$x=\sqrt{\frac{a\sqrt{(2g)}}{2\pi}}\ldots.(14),$$

$$x=\sqrt{\frac{a\sqrt{(2g)}}{2\pi}}.y^{-\frac{1}{4}}\ldots.(15),$$

respectively. The shape of the vessel in case of square-roots is therefore cylindrical, and if the radius of the cylinder is made according to (14) *the time of emptying the cylindrical vessel will be equal to the square root of the original depth of water.*

The physical conditions of the problem make it clear that the first method is more accurate. The determination of the time in the second method is liable to be affected with an error of a higher order than those occuring in the statical extraction of a root.

In the near future the author shall publish a remarkably simple extension of the first method to the solution of an equation of the form

$$a_0+a_1x+a_2x^2+a_3x^3+\ldots.+a_nx^n=0.$$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

133. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

In Wentworth's Arithmetic he gives a formula $\frac{2}{4}\frac{1}{0}(d^2-2d)$ for calculating the number of board feet in a log 10 feet long, when d is the diameter in inches. How is this rule derived?

Solution by P. H. PHILBRICK, C. E., Lake Charles, La.

The number of board feet in a log d inches in diameter and l inches in length is $n=\frac{\frac{1}{4}\pi d^2 l}{144}$.

If l is in feet, $n=\frac{\frac{1}{4}\pi d^2 l}{12}$.

Substituting $\frac{22}{7}$ for π and 10 for l we have, $n=\frac{5}{6}.\frac{1}{14}d^2$.

Allowing $\frac{1}{5}$ for saw cut, $n=\frac{4}{5}.\frac{5}{6}.\frac{1}{14}d^2=\frac{1}{21}d^2$.

Allowing $\frac{1}{2}$ inch for bark, $n=\frac{1}{21}(d-1)^2$.

For $d=22$, an average value, $(d-1)^2=\frac{441}{440}(d^2-2d)$.

$\therefore\ n=\frac{1}{21}.\frac{441}{440}(d^2-2d)=\frac{21}{440}(d^2-2d)$.

I would propose the formula $n=d^2$ (or $n=(d-1)^2$) for a log 20 feet long, since it is as accurate and much more simple than Wentworth's.

The most accurate formula, however, must be based on the end diameters of the log.

Let d and D represent those diameters.

Board feet in total volume of log 20 feet long

$$=\tfrac{1}{4}\pi.\tfrac{5}{3}\times\frac{d^2+dD+D^2}{3}=\tfrac{5}{6}\times\tfrac{11}{14}(d^2+dD+D^2).$$

(See Philbrick's *Engineer's Manual*, table 23).

Since $(D-d)^2=D^2-2dD+d^2>0$, $D^2+dD+d^2>3dD$.

Hence volume$>\frac{5}{3}\times\frac{11}{14}dD$.

Allowing $\frac{1}{5}$ for saw cut, $n>\frac{4}{3}\times\frac{11}{14}dD=\frac{22}{21}dD$.

Allowing $\frac{1}{2}\frac{1}{2}$ of the above for bark we still have $n>dD$, or, say, $n=dD$ ....(1).

It is easily shown that volume$>\frac{5}{3}\times\frac{11}{14}\left[\frac{d+D}{2}\right]^2$

and as before that $n=\left[\frac{d+D}{2}\right]^2$....(2).

The author's experience leads him to believe that the above formulas are quite accurate, but that logs will cut a little more into large timbers than the formulas give.

If thought to give too large a result, in extreme cases, we might suggest the formula, $n=d(D-2)$....(3), or $n=\left[\frac{d+D}{2}-1\right]^2$....(4).

At all events the forms suggested should be used.

ALGEBRA.

---

111. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL.D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Solve the equation $x(y+z)=a(x+y+z)$, $y(x+z)=b(x+y+z)$, $z(x+y)=c(x+y+z)$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Let $x+y+z=s$. Then (1)+(2)+(3) gives $xy+xz+yz=\frac{1}{2}(a+b+c)s$..(4).

(4)−(1) gives $yz=\frac{1}{2}(b+c-a)s$....(5).

(4)−(2) gives $xz=\frac{1}{2}(a+c-b)s$....(6).

(4)−(3) gives $xy=\frac{1}{2}(a+b-c)s$....(7).

(6)÷(7) gives $z/y=(a+c-b)/(a+b-c)$....(8).

(8) multiplied by (5) gives $z=\sqrt{\frac{(a+c-b)(b+c-a)s}{2(a+b-c)}}$....(9).

Similarly, $y=\sqrt{\frac{(a+b-c)(b+c-a)s}{2(a+c-b)}}$, $x=\sqrt{\frac{(a+b-c)(a+c-b)s}{2(b+c-a)}}$....(10, 11).

(9)+(10)+(11) gives $\sqrt{s}=\frac{2ab+2ac+2bc-a^2-b^2-c^2}{\sqrt{[2(a+b-c)(a+c-b)(b+c-a)]}}$....(12).

(12) in (9), (10), (11) gives

$$x(b+c-a)=y(a+c-b)=z(a+b-z)=\tfrac{1}{2}(2ab+2ac+2bc-a^2-b^2-c^2).$$

Also solved by *J. M. BOORMAN, W. H. CARTER, C. C. CROSS, LESLIE L. LOCKE, COOPER D. SCHMITT, ELMER SCHUYLER, J. SCHEFFER, B. F. YANNEY, J. W. YOUNG*, and *M. A. GRUBER*.

GEOMETRY.

138. Proposed by JOHN M. HOWIE, Professor of Mathematics, The Nebraska State Normal, Peru, Neb.

K is the middle point of any chord AB of a given circle. CD and EF are any two chords passing through K. CF and ED intersect AB at M and N, respectively. Prove that KM equals KN.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.; P. S. BERG, B. Sc., Larimore, N. D.; and F. B. FILLMAN, Chester, Pa.

CASE I. N and M within the circle.

$\triangle DKN/\triangle CKM=DK.NK/CK.MK$....(1).

$\triangle EKN/\triangle FKM=EK.NK/FK.MK$....(2).

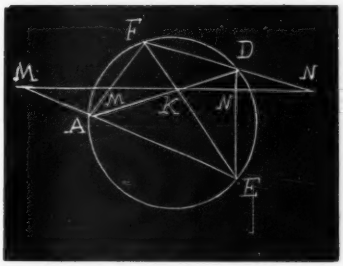
$\triangle CKM/\triangle EKN=CM.CK/EN.EK$....(3).

Multiplying (1) by (3),

$$\frac{\triangle DKN}{\triangle EKN}=\frac{DK.NK.CM}{MK.EN.EK}=\frac{DN}{EN}....(4).$$

Multiplying (2) by (3),

$$\frac{\triangle CKM}{\triangle FKM}=\frac{NK.CM.CK}{EN.FK.MK}=\frac{CM}{FM}....(5).$$



From (4) and (5),

$NK/MK=DN.EK/DK.CM$....(6).

$NK/MK=EN.FK/CK.FM$....(7).

Multiplying (6) by (7),

$$\therefore \frac{NK^2}{MK^2}=\frac{DN.EN.EK.FK}{CM.FM.DK.CK}\ldots(8).$$

But $EK.FK=DK.CK=AK^2$.

$$DN.EN=BN.AN=(BK-KN)(BK+KN)=BK^2-KN^2.$$

$$CM.FM=AM.MB=(BK-MK)(BK+MK)=BK^2-KM^2.$$

$$\therefore \frac{NK^2}{MK^2}=\frac{BK^2-NK^2}{BK^2-MK^2}.$$

$\therefore NK^2.BK^2-NK^2.MK^2=MK^2.BK^2-NK^2.MK^2$.

$\therefore NK^2.BK^2=MK^2.BK^2$.

$\therefore NK=MK$.

Case II. N and M without the circle.

$\triangle DKN/\triangle CKM=DK.NK/CK.MK$....(9).

$\triangle FKN/\triangle EKM=FK.NK/EK.MK$....(10).

$\angle KDN$ is the supplement of $\angle FDK$ and therefore the supplement of $\angle KEC$.

$\therefore \triangle EKM/\triangle DKN=KE.ME/KD.ND$....(11).

Multiplying (9) by (11),

$$\frac{\triangle EKM}{\triangle CKM}=\frac{NK.KE.ME}{ND.CK.MK}=\frac{ME}{MC}\ldots(12).$$

Multiplying (10) by (11),

$$\frac{\triangle FKN}{\triangle DKN}=\frac{FK.NK.ME}{KD.ND.MK}=\frac{NF}{ND}\ldots(13).$$

From (12) and (13),

$NK/MK=ND.CK/MC.KE$....(14).

$NK/MK=NF.KD/FK.ME$....(15).

Multiplying (14) by (15),

$$\frac{NK^2}{MK^2}=\frac{ND.NF.CK.KD}{MC.ME.FK.KE}=\frac{ND.NF}{MC.ME}=\frac{NB.NA}{MB.MA}=\frac{(NK-BK)(NK+BK)}{(MK-BK)(MK+BK)}$$

$$= \frac{NK^2 - BK^2}{MK^2 - BK^2}.$$

$\therefore NK^2.MK^2 - NK^2.BK^2 = NK^2.MK^2 - MK^2.BK^2.$
$\therefore NK^2.BK^2 = MK^2.BK^2.$
$\therefore NK = MK.$

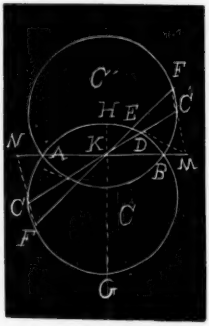
II. Solution by F. E. MILLER, A. M., Professor of Mathematics, Otterbein University, Westville, Ohio, and P. C. CULLEN, Indianola, Neb.

K the mid-point of chord AB, and CD and EF chords through K.

To prove that the joins CF and ED meet AB equidistant from K.

Through A and B draw circles $C' \equiv C$ and produce CD and EF to C' and F'.

From symmetry we see that FC and $F'C'$ meet AB equidistant from K and are parallel.



$\angle EDC = \angle EFC = \angle EF'C'$ and hence $EDC'F'$ are concyclic. Then is $F'C'$ and ED meet in M, $MD.ME = MC'.MF'$, or the tangents from M to the circles C and C' are equal and therefore M is on the radical axis, *i. e.* on AB.

$\therefore NK = MK.$

Again by projection.

Project circle C on a plane through AB so that the projection of HG perpendicular to AB may have the projection of K as its mid-point. Then the circle becomes an ellipse with K as center and CD and EF as diameters, and ED and FC meeting the major axis AB equally distant from the center K. But points on AB are not changed by the projection. Therefore N and M are always equidistant from K.

A second solution by Analytical Geometry was furnished by Professor Zerr.

CALCULUS.

104. Proposed by M. E. GRABER, Heidelberg University, Tiffin, Ohio.

Find the differential equation corresponding to $\sqrt{(1-x^2)} + \sqrt{(1-y^2)} = [a(x-y)]$.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tenn.; and the PROPOSER.

$\sqrt{(1-x^2)} + \sqrt{(1-y^2)} = a(x-y) \ldots . (1).$
$x dx/\sqrt{(1-x^2)} + y dy/\sqrt{(1-y^2)} = a(dy-dx) \ldots . (2).$

Eliminating a between (1) and (2) and reducing we get

$$\{xy - 1 - \sqrt{[(1-x^2)(1-y^2)]}\}\sqrt{[1-y^2]}\,dx$$
$$= \{xy - 1 - \sqrt{[(1-x^2)(1-y^2)]}\}\sqrt{[1-x^2]}\,dy.$$

$\therefore dx/dy = \sqrt{(1-x^2)}/\sqrt{(1-y^2)}.$

NOTE ON RADIUS OF CURVATURE.

By GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

The following method of deducing the formula for the radius of curvature seems to have some pedagogical advantages. The ordinary method where infinitesimals are used leaves some doubt in the mind of the beginner as to the exactness of the result. Especially is this true when the center of curvature is defined as the intersection of two consecutive normals.

Let (x_1, y_1), (x_2, y_2) be any two points of the curve, m_1, m_2 the slope of the tangents at these points. Then

$$x+m_1y=x_1+m_1y_1,$$
$$x+m_2y=x_2+m_2y_2,$$

are the equations of the normals. Solving these equations we find that the normals intersect at the point whose coördinates are

$$x=\frac{m_2x_1+m_1m_2y_1-m_1x_2-m_1m_2y_2}{m_2-m_1}=\frac{(m_1-m_2)x_2-m_2(x_1-x_2)-m_1m_2(y_1-y_2)}{m_1-m_2}$$

$$y=\frac{(x_1-x_2)+(m_1-m_2)y_1+m_2(y_1-y_2)}{m_1-m_2};$$

$$\text{or, } x=x_2-\frac{\frac{m_2}{m_1-m_2}}{\frac{x_1-x_2}{}}-\frac{m_1m_2\left(\frac{y_1-y_2}{x_1-x_2}\right)}{\frac{m_1-m_2}{x_1-x_2}},$$

$$y=y_1+\frac{1+m_2\frac{y_1-y_2}{x_1-x_2}}{\frac{m_1-m_2}{x_1-x_2}}.$$

When (x_2, y_2) approaches (x_1, y_1), we have

$$\lim(m_2)=m_1=\frac{dy_1}{dx_1};\ \lim\left(\frac{y_1-y_2}{x_1-x_2}\right)=\frac{dy_1}{dx_1};\ \lim\left(\frac{m_1-m_2}{x_1-x_2}\right)=\frac{d^2y_1}{dx_1^2}.$$

$$\text{Then } x-x_1=-\frac{dy_1}{dx_1}\left\{\frac{1+\left(\frac{dy_1}{dx_1}\right)^2}{\frac{d^2y_1}{dx_1^2}}\right\},\ y-y_1=\frac{1+\left(\frac{dy_1}{dx_1}\right)^2}{\frac{d^2y_1}{dx_1^2}}.$$

The distance of point of intersection from foot of normal is therefore

$$\rho=\frac{\left[1+\left(\frac{dy_1}{dx_1}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y_1}{dx_1^2}}.$$

The equation of a circle whose center is the point of intersection and which passes through the foot of the normal is

$$(x-x_1)^2+(y-y_1)^2=\rho^2.$$

Differentiating this twice and eliminating x, y, we find

$$\rho=\frac{\left[1+\left(\frac{dy_1}{dx_1}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y_1}{dx_1^2}}.$$

This shows that a circle having the same slope and same value of $\frac{d^2y}{dx^2}$ at its point of intersection with a given curve has its center at the limiting point of intersection of two normals.

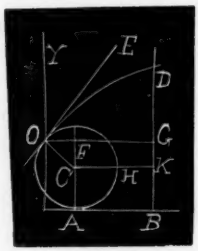
MECHANICS.

100. Proposed by WALTER H. DRANE, Graduate Student, Harvard University; Cambridge, Mass.

A man, riding a bicycle, runs through a puddle of water and a bit of mud is thrown from the rear wheel and alights on the crown of his hat. Supposing the wheel 28 inches in diameter, that the man's head is 6 feet above ground, that the saddle is 1 foot in front of the rear wheel, and that the mud left the wheel at a point 30° from highest point of wheel, how long will it take a man to ride a mile at this rate?

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and L. R. INGERSOLL, Student Colorado College, Colorado Springs, Col.

As the particle of mud and man have the same uniform velocity forward, it is not necessary to consider such motion. The result will be the same if we regard the man at rest and the hind wheel of the bicycle revolving with the same velocity as the man is moving forward. Let O be the origin of the coördinates, D the top of the man's head, $\angle EOG=\angle OCF=\theta$. OE the tangent to the wheel at O, $CO=a=14$ inches, $HK=12$ inches, $BD=72$ inches, $g=32.16$ feet. Then $y=x\tan\theta-gx^2/2v^2\cos^2\theta$, is the equation to OD, the path of the mud.



$$\therefore\ v=\frac{x}{2\cos\theta}\sqrt{\frac{2g}{x\tan\theta-y}}\ldots.(1).$$

$$x=OG=OF+FG=OF+CH+HK=26+14\sin\theta.$$
$$y=GD=BD-BK-KG=72-14-14\cos\theta=58-14\cos\theta.$$

$$\therefore\ v=\frac{(13+7\sin\theta)}{\cos\theta}\sqrt{\frac{g}{(13+7\sin\theta)\tan\theta+7\cos\theta-29}}.$$

When $\theta=30°$, $v=59\sqrt{-3}$ inches, an impossible result.

$\therefore$ $GD>$ than the intersection made by the particle on BD and indicates that the mud would never get 6 feet above the ground.

Let $\theta=60°$, $v=273.17$ inches$=22.76$ feet per second.

$t=5280\div22.76=231.98$ seconds$=3$ minutes, 51.98 seconds, time required to ride a mile.

101. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

Find the center of gravity of a cone that has a specific gravity of 1 (one) at the top and 2 (two) at the base.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; WILLIAM W. LANDIS, A. M., Dickinson College, Carlisle, Pa.; H. C. WHITAKER, Ph. D., Manual Training School, Philadelphia, Pa.

Let $y=m(x-a)$ be the equation to the generator of the cone.

$$\text{Then}\quad \bar{x}=\frac{\int\rho y^2x\,dx}{\int\rho y^2\,dx}=\frac{\int_a^{2a}\rho x(x-a)^2\,dx}{\int_a^{2a}\rho(x-a)^2\,dx}.$$

By the conditions of the problem, $\rho=x/a$.

$$\therefore\ \bar{x}=\frac{\int_a^{2a}x^2(x-a)^2\,dx}{\int_a^{2a}x(x-a)^2\,dx}=\frac{\frac{31a^5}{30}}{\frac{7a^4}{12}}=\frac{62a}{35}.$$

$\bar{y}=0$. $\quad\frac{62a}{35}-a=\frac{27a}{35}=$ the distance of the center of gravity from the vertex.

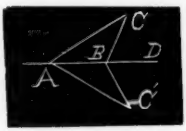
AVERAGE AND PROBABILITY.

70. Proposed by Professor MILLER.

A ship at A observes another at B, whose course is unknown. Supposing their speed the same, prove that the chance of their coming within a given distance, d, of each other is always $(2/\pi)\sin^{-1}(d/a)$, whatever the course taken by A; provided its inclination to AB is not greater than $\cos^{-1}(d/a)$, where $AB=a$. [From *Cambridge Mathematical Tripos*, 1871.]

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $AB=a$, $\angle CAB$ which ship A makes with $AB=\theta$, $\angle CBD$ which ship B makes with $AB=\varphi$ where C is the intersection of the two courses. Then $\angle ACB=(\varphi-\theta)$.



Let v=velocity of each ship, $AC=b$, $BC=c$.

Then in time t, the ship is distant from C, $b-vt$. B is distant from C, $c-vt$.

$\therefore\ d^2=(b-vt)^2+(c-vt)^2-2(b-vt)(c-vt)\cos(\varphi-\theta)$ ....(1).

Differentiating with reference to t for a minimum, we get

$$v(b-vt)+v(c-vt)=[v(c-vt)+v(b-vt)]\cos(\varphi-\theta).$$

$\therefore\ t=(b+c)/2v$....(2).

Substituting (2) in (1) we get $d=(b-c)\cos\frac{1}{2}(\varphi-\theta)$.

But $b=a\sin\varphi/\sin(\varphi-\theta)$, $c=a\sin\theta/\sin(\varphi-\theta)$.

$$\therefore\ d=\frac{a\cos\frac{1}{2}(\varphi-\theta)(\sin\varphi-\sin\theta)}{\sin(\varphi-\theta)}.$$

Now $\sin\varphi-\sin\theta=2\cos\frac{1}{2}(\varphi+\theta)\sin\frac{1}{2}(\varphi-\theta)$.

$\therefore\ d=a\cos\frac{1}{2}(\varphi+\theta)$.

$\therefore\ \varphi=2\cos^{-1}(d/a)-\theta=\theta_1$.

Let $\cos^{-1}(d/a)=\beta$.

$$\therefore\ \text{Chance}=\frac{\int_{-\beta}^{\beta}\int_{\theta_1}^{2\pi-\theta_1}d\theta d\varphi}{\int_{-\beta}^{\beta}\int_{0}^{2\pi}d\theta d\varphi}=\frac{\int_{-\beta}^{\beta}(2\pi-2\theta_1)d\theta}{4\pi\beta}=\frac{(\pi-2\beta)}{\pi}.$$

$\therefore\ p=(2/\pi)[\frac{1}{2}\pi-\cos^{-1}(d/a)]=(2/\pi)\sin^{-1}(d/a)$.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

138. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

"A pound of gold may be drawn into a wire that would extend around the earth." What would be the diameter of such a wire if the specific gravity of gold is 19.36 and the distance is 24,900 miles?

139. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

The *ratio* of the interest to the true discount on a certain principal for a certain time at a certain rate per cent. per annum, is $m=21$ to $n=20$. What is the rate per cent.?

*** Solutions of these problems should be sent to B. F. Finkel not later than March 10.

ALGEBRA.

127. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Sum to n terms the series

$$\frac{4}{1.2.3}\cdot\frac{1}{3}+\frac{5}{2.3.4}\cdot\frac{1}{3^2}+\frac{6}{3.4.5}\cdot\frac{1}{3^3}+\ldots.$$

128. Proposed by ELMER SCHUYLER, B. Sc., Teacher of German and Mathematics, Boys' High School, Reading, Pa.

Solve $(1+x^3)(1+x^2)(1+x)=30x^3$.

129. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

Prove $f(x)=e^{e^x}=e+\sum_{r=1}^{r=\infty} a_r x^r$, where $a^r=\frac{1}{r\,!}\sum_{\kappa=1}^{\kappa=\infty}\frac{\kappa^r}{\kappa\,!}$.

*** Solutions of these problems should be sent to J. M. Colaw not later than March 10.

GEOMETRY.

157. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Find the locus of the center of a circle touching a given line and always passing through a given point.

158. Proposed by JOHN MACNIE, A. M., Professor of Latin, University of North Dakota.

Show by a simple diagram that:

(*a*) If the angle-sum of an equilateral triangle is constant, that constant is a straight angle.

(*b*) If the angle-sum is less than a straight angle, the sum increases as the triangle grows less.

(*c*) If the angle-sum is greater than a straight angle, the sum decreases as the triangle grows less.

159. Proposed by FRANCIS W. HANAWALT, Professor of Mathematics and Astronomy, Iowa Wesleyan University, Mt. Pleasant, Iowa.

A man desires to lay out a half mile race course by using two circles of 150 feet radius and their internal tangents. How far apart shall the circles be placed?

*** Solutions of these problems should be sent to B. F. Finkel not later than March 10.

CALCULUS.

119. Proposed by JOHN M. COLAW. A. M., Monterey, Va.

Rectify the Folium of Descartes, the equation of which is $x^3+y^3+3axy=0$.

120. Proposed by G. B. M. ZERR. A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The axis of a paraboloid of revolution coincides with the generating line of a cylinder; the diameter of the cylinder and the latus-rectum of the parabola are each equal to the common altitude, a. Find the surface and volume of each part into which the paraboloid is divided by the cylinder.

121. Proposed by W. W. LANDIS, A. M., Professor of Mathematics and Astronomy, Dickinson College, Carlisle, Pa.

Solve the differential equation $\left[\frac{d}{dx}+b\right]^n y=\cos ax$.

122. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Solve the differential equation $(y-x)\sqrt{(1+x^2)}\frac{dy}{dx}=n(1+y^2)^{\frac{1}{2}}$.

123. Prize Problem. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Find in finite terms, the value of $\int_0^{\frac{1}{4}\pi}\tan\phi d\phi$.

A year's subscription to the MONTHLY will be given to the person sending to the Proposer the first solution of this problem.

*** Solutions of these problems should be sent to J. M. Colaw not later thah March 10.

MECHANICS.

107. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

A rough uniform rod, length $2a$, is placed with a length $c(>a)$ projecting over the edge of the table. Prove that the rod will begin to slide over the edge when it has turned through an angle $\tan^{-1}\left[\frac{\mu a^2}{a^2+9(c-a)^2}\right]$.

[From Loudon's *Rigid Dynamics*.]

108. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Prove that the *inclination* of a perfectly rough inclined plane must be $\theta=\sin^{-1}[e^2/(2-e^2)]$, in order that an ellipse of minimum eccentricity e may be capable of resting in equilibrium on the plane.

*** Solutions of these problems should be sent to B. F. Finkel not later than March 10,

AVERAGE AND PROBABILITY.

99. Proposed by E. B. SEITZ.

A point is taken at random in the surface of a given circle, and from it a line equal in length to the radius is drawn, so as to lie wholly in the surface of the circle. Find the chance that the line intersects in a given diameter. [No. 135, *The Analyst.*]

100. Proposed by L. C. WALKER, Associate Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Required the average distance between two points in opposite sides of a regular $2n$—gon.

*** Solutions of these problems should be sent to B. F. Finkel not later than March 10.

MISCELLANEOUS.

100. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Determine the maximum value of $(\varphi-\varphi')$, if given electric currents C and C' produce deflections φ and φ' in a tangent galvanometer, so that $\tan\varphi/\tan\varphi' = C/C'$.

101. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A wire is laid along the surface of a right cone semi-vertical angle β so that it cuts the generators everywhere at a constant angle θ. Find the radius of curvature and radius of torsion.

*** Solutions of these problems should be sent to J. M. Colaw not later than March 10.

EDITORIAL NOTE.

The MONTHLY begins the twentieth century with eighth volume. Seven volumes are already completed, and we trust that by the coöperation of its friends it may complete many more volumes.

It is desirable to publish more papers in the future, but only such as are of real value and merit. It is not desirable to publish articles which are mere

modifications of subjects treated in the ordinary text-book, but translations of important subjects from the best European writers would be of real value to many of our readers. For example, a translation of Abel's, or Wantzel's Demonstration of the Impossibility of Resolving Algebraically General Equations of a Degree Higher than the Fourth, Hermite's proof that π is transcendental, subjects that are of great interest to teachers of mathematics. Careful translations of such and other subjects will be welcomed both by readers and the editors.

Through the kindness of a very warm friend of the MONTHLY, the expense of its publication for the past year was nearly all met without drawing painfully on the editors, and we trust that all our subscribers will remain with us and support us in our effort to make the MONTHLY indispensable to all teachers of mathematics.

BOOKS.

The Elements of Analytic Geometry. By Albert L. Candy, Ph. D., Adjunct Professor of Mathematics in the University of Nebraska. 8vo. Cloth, 303 pages. Published by the Author, Lincoln, Neb.

This book has many commendable features, among which are to be noticed: The correlation of subjects; the graphic treatment of the Theory of Equations connects it with Algebra; while Chapter V and the sections on Quadrature, Maxima and Minima, introduce to the student the fundamental notions of the Calculus. Numerous exercises in the earlier chapters involving trigonometric work will refresh the student's mind on that subject. The Theory of Parameter Coordinates is briefly but clearly presented. Many historical notes are inserted. In these ways, the book marks a departure from the ordinary texts on the subject. B. F. F.

Field Manual for Engineers. By Phileteus H. Philbrick, C. E., M. S., Member American Mathematical Society, Member American Society of Civil Engineers, Chief Engineer K. C. W. & G. R. R.

The aim of this book the author says, is, 1. To present the subject of the text in a mathematical and logical order. 2. To classify all problems presented so as to be easily referred to. 3. To express the resulting formula of every problem in the form requiring the least numerical computation. 4. To furnish a large number of useful tables, more complete, more extended, and, when possible, with more elementary and appropriate argument than other similar tables. 5. To treat the general problems of railway engineering more extensively than other similar works have done. After having examined the work we believe that the author has accomplished what he aimed to do. He has written a work which is up to the demands made by the march of modern railway systems of the past twenty years. B. F. F.

ERRATA.

In Problem 153, Geometry, page 234, second line, for "the point" read *a focus*.

Page 299, line 12 from top, in denominator of $\tan^{-1}$ insert $-c^2$.

Page 302, in the notice of Prof. Nichols' Calculus, the publishers should be D. C. Heath & Co., instead of Allyn & Bacon.

KARL FREDERICH GAUSS.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. VIII. FEBRUARY, 1901. No. 2.

BIOGRAPHY.

KARL FREDERICH GAUSS.

BY B. F. FINKEL.

This versatile and prolific mathematician, Karl Frederich Gauss, was born at Brunswick, Germany, April 30*, 1777, and died at Göttingen on February 23, 1855. His father was a brick-layer and was desirous of profiting by the wages of his son as a laborer, but young Gauss's talents attracted the attention of Bartels, afterwards professor of mathematics at Dorpat, who brought him to the notice of Charles William, Duke of Brunswick. The duke undertook to educate the boy and sent him to the Caroline college, in 1792. By 1795 it was admitted alike by professors and pupils that he knew all that the professors could teach him- It was while at this school that he investigated the method of least squares, and proved by induction the Law of Quadratic Reciprocity. He gave the first rigorous proof of this law and succeeded in discovering eight different demonstrations of it.† While at Caroline college, Gauss manifested as great an aptitude for language as for mathematics, a very general characteristic of eminent mathematicians.

In 1795 Gauss went to Göttingen, as yet undecided whether to pursue philology or mathematics. While at Göttingen he studied mathematics under

*Cf. *Britannica Encyclopedia* and *Century Dictionary*.

†For Gauss's third proof as modified by Dirichlet, see Mathews's *Theory of Numbers*, pages 38-41.

Abraham Gotthelf Kästner, who was not a very inspiring teacher and who is now chiefly remembered for his *History of Mathematics*, 1796, and by the fact that he was a teacher of the illustrious Gauss. In 1796 he discovered a method of inscribing in a circle a polygon of seventeen sides, and it was this discovery that encouraged him to pursue mathematics rather than philology—a rather insignificant incident to be fraught with such stupendous consequences—consequences materially affecting our present progress in mental and material development.

A detailed construction of this problem by elementary geometry was first made by Pauker and Erchinger.

Gauss worked quite independently of his teachers at Göttingen, and it was while he was there as a student that he made many of his greatest discoveries in the theory of numbers, his favorite subject of investigation. Among his small circle of intimate friends was Wolfgan Bolyai, the discoverer of non-Euclidean geometry.

In 1798 Gauss returned to Brunswick, where he earned a livelihood by private tuition. Later in the year he repaired to the University of Helmstadt to consult the library, and it was while here that he made the acquaintance of Pfaff, a mathematician of great power. Laplace, when asked who was the greatest mathematician in Germany, replied, Pfaff. When the questioner said he should have thought Gauss was, Laplace replied: "Pfaff is the greatest mathematician in Germany; but Gauss is the greatest in all Europe."*

In 1799 Gauss published his demonstration that every algebraical equation with integral coefficients has a root of the form $a+bi$, a theorem of which he gave three distinct proofs. In 1801, he published *Disquisitiones Arithmeticæ*, a work which revolutionized the whole theory of numbers. "The greater part of this most important work was sent to the French Academy the preceding year, and had been rejected with a sneer which, even if the work had been as worthless as the referees believed, would have been unjustifiable."† Gauss had written far in advance of the judges of his work, and so the recognition of its merits had to wait until the mathematical world came in sight of this splendid creation. Gauss was deeply hurt because of this unfortunate incident; and it was partly due to it that he was so reluctant to publish his subsequent investigations.

The next important discovery of Gauss was in a totally different department of mathematics. The absence of a planet between Mars and Jupiter, where Bode's Law would have led observers to expect one, had long been remarked, but not until 1801 was any of the numerous groups of minor planets which occupy that space observed. On the first of January, 1801, Piazzi of Palermo discovered the first of these planets, which he called Ceres, after the tutelary goddess of Sicily.‡ While the announcement of this discovery created no great surprise, yet it was very interesting, since it occurred simultaneously

*Cajori's *A History of Mathematics.*
†Ball's *A Short History of Mathematics.*
‡Young's *General Astronomy*, edition of 1898, page 368.